On the Convergence of Puck Clustering Systems

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Abstract

Puck clustering involves the physical relocation of small objects known as 'pucks' from random positions to a central location, which need not be specified *a priori*. The evolution of systems of clusters of pucks under the action of robots capable of moving pucks from several locations is considered. A general set of conditions by which a puck collection system may be seen to evolve to a one-cluster system is developed. General conditions leading to clustering behavior in the presence and absence of non-embodied agents are derived. Conditions leading to more efficient algorithms are also derived. Several examples are given to illustrate both non-embodied and embodied puck clustering systems.

keywords: swarm engineering, puck clustering

1 Minimalist engineering: puck clustering

Puck clustering is a decentralized action taken by one or more agents in which objects, which are originally arranged in one or more groupings, are manipulated into a predetermined number of clusters of predetermined size. This, of course, is easily accomplished using rather well-equipped agents which have a relatively accurate global picture of the system. However, accomplishing this with simple robots with a partial and, in most cases, extremely limited knowledge of the system is a rather daunting task on the face of it. The difficulty comes from predicting the detailed behavior of the group of robots and its complex interaction with the elements of the system. In general, this is an unsolved problem. However, solutions to this problem in even cases of somewhat restricted complexity can yield behaviors of rather significant interest, and may lead to methods of accomplishing more interesting behaviors, such as construction behaviors.

Puck clustering teams are examples of groups of robots arising from work done in the realm of minimalist design using robot groups and simple actuators and sensors[7]. These systems consist of robots which have very simple controllers including a small amount of memory (if any at all) and rigid hierarchical controllers. Proposed initially by Brooks [3], this view has gained increasing support as ever more interesting systems are designed using robots of this design paradigm. The allure of these systems stems from the ability to use very simple robots to achieve complex tasks and obtain behavior that is similar to some intelligent behavior.

Initial biological studies of insect systems [6][8] indicated that ants and termites are creatures which utilize rather rigid and simple behaviors and carry out somewhat simple algorithms. Despite the fact that the behaviors utilized by ants and termites can be extremely rigid, their nests can be extremely efficient, flexible, and adaptive. Work on systems designed to take advantage of this newly discovered emergent property of systems of rigid agents has uncovered interesting design paradigms governing possible construction techniques and optimal resource exploitation algorithms [2][4][5][11]. These studies indicated that simple rules and stigmergic actions were all that were required for the construction of complex structures. This formed some of the motivation behind the robotic test cases in which the first construction task – puck clustering – is undertaken.

One of the earliest attempts to design puck clustering robots came from Beckers et. al. [1]. In this study, a group of robots is equipped with a curved passive gripper in the front of the robot. This gripper is capable of capturing "pucks" and holding them as the robot moves. When three or more are being pushed by the gripper, it triggers a backup motion, which leaves the pucks where they are. The long time behavior of such a system produces a single cluster containing all pucks not pushed by the robot. Average simulations of three robots take just over an hour and a half to complete. In the final stable state, a single cluster is formed. The system is capable of forming the same stable end state with only one robot, but this takes a concommittant longer time.

Maris and Boekhorst [9] present a system that is similar to that of Beckers et. al. However, in this study, the robots do not have grippers, and so cannot recover pucks that have been pushed to a boundary. In this system, pucks are pushed until a barrier (which may be another puck) is encountered, at which time the puck is left behind and the robot turns and goes in another direction. Once a cluster is big enough it becomes stable, as the robot cannot any longer approach the cluster and take off pucks. In this study, many clusters of size one or two were created near the boundaries, though other larger clusters were found in the interior of the arena. There was no global cluster formed, though one or two large ones were typically formed.

The main limitation of these studies is the lack of applicability. Once we understand how these systems work, there is no indication that our understanding can be directly applied to other related systems. Investigation of the new systems require as much effort after the initial investigation as it would have without the investigation. This is a serious detriment to our design capabilities, and would seem to relegate these studies to the realm of anecdotal evidence on the same level as that obtained by insect studies.

In this paper, we take a first step in developing a general theory of clustering and construction systems. We develop a formalism for clustering systems based first on what we call *aphysical agents*. This formalism generates a condition which must be satisfied in order for clustering to occur. Once this has been accomplished, we develop a second formalism that is appropriate for real physical systems. This yields another condition which turns out to be identical to the first condition. Once this has been accomplished, we establish methods of examining the efficiency of differing designs. Next, we examine the question of how to actually build robot systems that generate the desired global behavior, and examine two existing designs from the literature. Finally, we take a closer look at the condition developed early on and demonstrate how alterations of this condition can lead to more interesting clustering systems with clusters of differing and predetermined sizes. The last Section offers a few concluding remarks.

2 A simple aphysical system

Before attacking the general problem of building physical robots capable of moving in a real environment and carrying out the clustering task, we consider the rather trivial problem of determining the conditions under which we might expect *aphysical robots*, robots whose properties cannot possibly be commensurate with a physical implementation, to create a single cluster of pucks. For this purpose we can then consider clusters to be nothing more than large collections of pucks with no physical form or implementation. We also consider the robots to be nothing more than agents capable of picking up, carrying, and depositing pucks. The robot need not have physical form, and its interaction with the cluster need not have physical meaning.

We approach the problem in this way for a rather simple reason. Many constraints are placed on realistic robotics by embodiment. However, at the simplest level, puck clustering need not be restricted to embodied robots, and the general theory might find applications elsewhere. Therefore, we begin with the simplest system, and draw conclusions about the simplest conditions required for clustering to occur. In later sections, we will discuss the implications of embodiment and how one might connect this general theory to real physical robots.

For the purposes of this Section, we assume that the system may be approximated by an iterative process whereby at each interval in time each robot interacts with a randomly chosen cluster. The cluster chosen will have some number of pucks contained within, and the robot's interaction will depend on whether or not it is currently carrying a puck. At each iteration, each robot randomly chooses a cluster to interact with, carries out its interaction, and then waits for all other robots to interact.

2.1 Two cluster system

We consider a system of two clusters C_1 and C_2 and n_c robots. We assume that cluster C_1 contains n_1 pucks and that cluster C_2 contains n_2 pucks. The robots, as discussed above, have no particular physical structure or rules of motion. Rather they can interact with either of the clusters at will. This allows us to think of the robots as a medium in which the clusters exist.

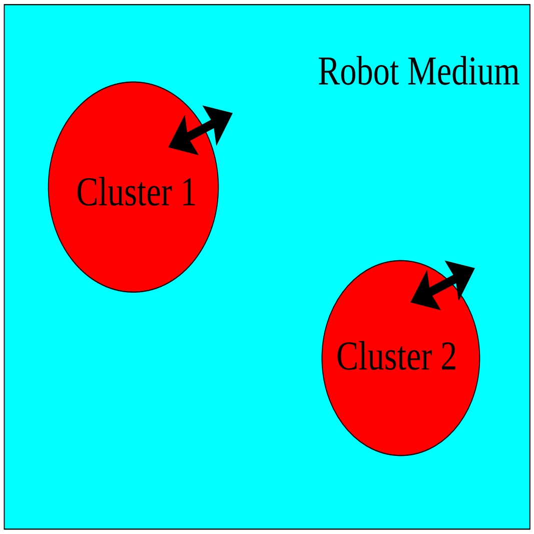


Figure 2.1.1: This is an abstract view of a puck-clustering system. In this view, the clusters are effectively reservoirs of pucks, and the robots are simply pathways for pucks to move between clusters.

If the number of robots holding a puck is n_m then, under the assumption that a robot may only carry a single puck, it follows that the number of robots available to remove a puck from the cluster is $n_c - n_m$. The rate of change of pucks in the two clusters C_1 and C_2 can be written

$$\frac{dn_1}{dt} = -\left(n_c - n_m\right) f\left(n_1\right) + h\left(n_1\right) n_m \tag{1}$$

$$\frac{dn_2}{dt} = -\left(n_c - n_m\right) f\left(n_2\right) + h\left(n_2\right) n_m \tag{2}$$

where, f is the likelihood of puck removal and h is the likelihood of puck deposit.

If it is assumed that the number of pucks in the transport media is stationary (i.e. on average $\frac{dn_m}{dt} = 0$), it then follows that

$$\frac{dn_1}{dt} = -\frac{dn_2}{dt}, \tag{3}$$

From (1), (2), and (3) we obtain

$$(n_c - n_m) f(n_2) - h(n_2) n_m = -(n_c - n_m) f(n_1) + h(n_1) n_m \ . \tag{4}$$

Solving for n_m gives

$$n_m = \frac{n_c (f(n_1) + f(n_2))}{f(n_1) + f(n_2) + h(n_1) + h(n_2)}. \tag{5}$$

This can then be substituted into (1) to produce the expression

$$\frac{dn_1}{dt} = n_c \frac{h(n_1) f(n_2) - h(n_2) f(n_1)}{f(n_1) + f(n_2) + h(n_1) + h(n_2)} \tag{6}$$

with an analogous expression for n_2.

The condition required for C_1 to grow given that $n_1 > n_2$ may be written as

$$\frac{dn_1}{dt} > 0; \ \forall \, n_1 > n_2. \tag{7}$$

If $h, \ f > 0$ then it follows from (6) that this condition is satisfied if

$$h(n_1) f(n_2) > h(n_2) f(n_1) \, . \tag{8}$$

and so

$$\frac{f(n_2)}{h(n_2)} > \frac{f(n_1)}{h(n_1)}, \tag{9}$$

Thus for C_1 to continue to grow

$$g(n_2) > g(n_1) \, ; \ \forall \, n_1 > n_2 \tag{10}$$

where $g(n) = \frac{f(n)}{h(n)}$. This condition holds if $g(n)$ is strongly monotonically decreasing and positive. Figure 2.1.2 shows an adequate form for g.

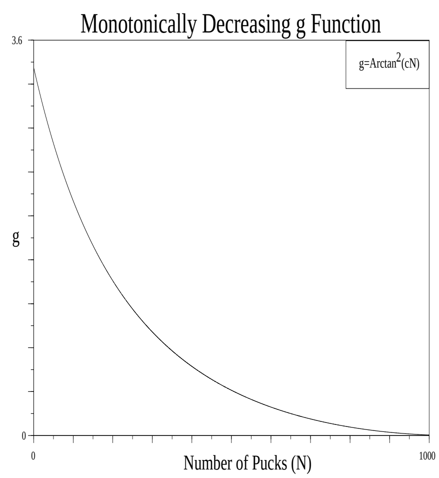


Figure 2.1.2: Any monotonically decreasing function will serve as a clustering condition.

Under the above assumptions, the condition for the growth of the largest cluster is that the ratio of f and h be a strongly monotonically decreasing function.

2.2 Multiple clusters

The previous subsection demonstrated the conditions under which the agents might be expected to create a single cluster as a result of their interactions with two clusters. This required the agents to have interactions with the clusters that were characterized by a strictly monotonically decreasing function g of the number of clusters. In this sub-section, we extend these results to multiple clusters.

Again, the puck cluster system is as defined above. The question of whether or not a particular cluster will shrink or grow depends on its interaction with the other clusters which may absorb evaporated pucks, as well as produce free pucks which may be collected by the cluster in question. This problem may be approached by deriving conditions under which any given cluster will monotonically decrease in size.

The density of pucks in the transport media is again assumed to be stationary. We consider the robots to be the same automatons as in the last sub-section. Then it must be that for the various clusters,

$$\frac{dn_1}{dt} = -\left(n_c - n_m\right) f\left(n_1\right) + h\left(n_1\right) n_m \tag{11}$$

$$\frac{dn_2}{dt} = -\left(n_c - n_m\right) f\left(n_2\right) + h\left(n_2\right) n_m \tag{12}$$

$$\vdots$$

$$\frac{dn_q}{dt} = -\left(n_c - n_m\right) f\left(n_q\right) + h\left(n_q\right) n_m \tag{13}$$

and

$$\frac{dn_1}{dt} + \frac{dn_2}{dt} + \cdots + \frac{dn_q}{dt} = 0 \tag{14}$$

This gives

$$n_m = \frac{n_c \sum_i f\left(n_i\right)}{\sum_i f\left(n_i\right) + \sum_i h\left(n_i\right)} \tag{15}$$

which leads to the equation:

$$\frac{dn_q}{dt} = \frac{n_c \sum_{i \neq q} \left(h\left(n_q\right) f\left(n_i\right) - f\left(n_q\right) h\left(n_i\right)\right)}{\sum_i f\left(n_i\right) + \sum_i h\left(n_i\right)} \tag{16}$$

The rate of change in the size of the cluster C_q will be negative provided

$$\frac{dn_q}{dt} < 0$$

which leads to the condition that

$$g\left(n_q\right) = \frac{f\left(n_q\right)}{h\left(n_q\right)} > \frac{\sum_{i \neq q} f\left(n_i\right)}{\sum_{i \neq q} h\left(n_i\right)} \equiv g(n_{eff}) \tag{17}$$

where $g(n_{eff})$ is the effective cluster of size n_{eff} equivalent to the interactions of all clusters other than q. In general, the determination of the effective rate must depend on the number of clusters and their relative sizes (as well as, eventually, their accessibility to the cluster in question).

Condition (17) is somewhat ambiguous, as it is not clear that $g(n_{eff})$ is actually smaller than any given $g(n_q)$. This is somewhat unsatisfying, so we consider the fate of the smallest cluster. Recall that if g is a monotonically decreasing function of cluster size then, for $n_1 > n_2$

$$f(n_2)\,h(n_1) > f(n_1)\,h(n_2) \ . \tag{18}$$

If C_q is the smallest cluster then

$$f(n_q)\,h(n_i) > f(n_i)\,h(n_q) \tag{19}$$

$\forall\ i \neq q$. Summing over all $i \neq q$ it follows that

$$\sum_{i\neq q} f(n_q)\,h(n_i) > \sum_{i\neq q} f(n_i)\,h(n_q) \ . \tag{20}$$

Rearranging, we find that

$$\frac{f(n_q)}{h(n_q)} > \frac{\sum_{i\neq q} f(n_i)}{\sum_{i\neq q} h(n_i)} \tag{21}$$

which is the condition for

$$\frac{dn_q}{dt} < 0 \tag{22}$$

Thus, irrespective of the forms of f and h, the time averaged behavior of the smallest cluster will be to decrease in size, as long as g is monotonically decreasing.

2.3 Induced carrier densities

It is interesting to ask about the average number of robots carrying occupancy of the robots in a more rigorous way. Previously, we assumed that the number of pucks being carried by the swarm was approximately constant. Now we ask just how realistic this assumption is, and what its effect is on the evolution of the system.

First, we consider the case in which our swarm is interacting with a single cluster. In this case, the cluster has some number of pucks, which we will denote as N. If we again define the number of agents as n_c and the number of agents carrying pucks as n_m then equation (1) gives us

$$\frac{dN}{dt} = -(n_c - n_m)\,f(N) + h(N)\,n_m \ .$$

In equilibrium, this will be zero, which gives us

$$(n_c - n_m)\,f(N) = h(N)\,n_m \ . \tag{23}$$

Solving for n_m gives

$$n_m = n_c \frac{f(N)}{f(N) + h(N)} = n_c \frac{1}{1 + \frac{1}{g(N)}} \ . \tag{24}$$

Thus, if g is a monotonically decreasing function, so too is n_m. However, since g must only be monotonically decreasing the extent to which n_m changes is limited. If $g \gg 1$ then $n_m \simeq n_c$. However, if $g \ll 1$ then $n_m \ll n_c$. In both cases, the variation in the number of pucks held by agents is small.

In order to have a firmer understanding of the link between n_m and g, we note that

$$\frac{dn_m}{dN} = n_c \frac{1}{(g+1)^2} \frac{dg}{dN} \tag{25}$$

Again, we can see that if $g \gg 1$ then the variation of n_m given by

$$\frac{dn_m}{dN} \simeq n_c \frac{1}{g^2} \frac{dg}{dN} \tag{26}$$

is quite small. However, if $g \simeq 0$ then

$$\frac{dn_m}{dN} = n_c \frac{dg}{dN} \tag{27}$$

which is also quite small. Interestingly, since $\frac{dg}{dN} < 0$ for puck clustering systems, we may expect that the number of pucks carried by robots is also a monotonically decreasing number.

Example of Number of Pucks

Suppose that we have two cross sections f and h given by

$$f = \sigma N^\alpha \text{ and } h = 1. \tag{28}$$

These together give us that

$$G = \sigma N^\alpha \ . \tag{29}$$

We may use this in our expression of n_m . We have in a discrete model

$$n_m = n_c \frac{1}{1 + \frac{h}{f}} = n_c \frac{\sigma N^\alpha}{\sigma N^\alpha + 1} \ . \tag{30}$$

In Figure 2.3.1, we plot the prediction of this model and the measured occupancy of the robots for a nonembodied single cluster system with a varying number of pucks and 150 robots.

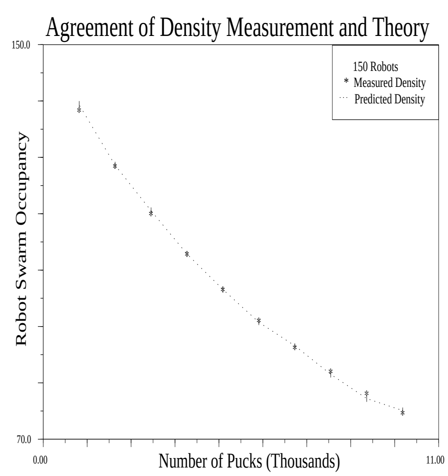


Figure 2.3.1: This figure gives the agreement of the predicted and measured values of the occupancy of robots in a swarm model. In this case, there is one cluster of varying size, the robots are nonembodied and quickly circulating, there are 150 robots, and the propensities are given by $f = \sigma N^\alpha$ and $h = 1$.

In Section 3, we will investigate an analogous quantity based on densities. We will find that the expressions are identical, leading to the same general rule.

2.4 Summation

This section has considered the totally unrealistic system of robots and clusters in which robots are considered to be agents which have the capability to randomly choose clusters and interact with them. The robots' interactions are dictated by their state which consists entirely of whether or not the robots are carrying a puck. The interactions depend only on whether or not the robot will change its state, hence picking up or dropping off a puck depending on its estimate of the state of the cluster. In the highest abstraction, this system accurately describes a puck clustering system. The details of the interaction such as the existence of a boundary, interference of other robots, inaccessibility of parts of clusters, etc., would seem to be part of the interaction. Thus, without making any specific predictions of a single system, we assert that the interactions, which generally must include these other concerns, are likely to require compliance with the same requirements as the interactions in this Section.

In the next Section, we consider the more rigid question of systems in which robots may not freely move about. We consider the behavior and interaction of the pucks moving about 'in the medium' under the influence of robots, and pucks currently captured by clusters. This will lead to a new set of conditions, which are comparable to those discovered in this system.

3 Limited accessibility

3.1 General considerations

In the last Section, we derived the general condition required for the interaction between a group of probabilistic agents and groups of undefined items called pucks to generate a single group of pucks. That solution was based on the assumption that the automatons had access to all the pucks in the cluster and access to all clusters. That assumption, of course, is not valid in real systems. Real robots cannot gain access to all the pucks in a given cluster, nor can they reach each cluster at will. In this Section, we derive the condition under which a set of clusters under the influence of one or more robots will monotonically move to a single cluster.

In a realistic experiment, robots are physically limited by the closed arenas in which they operate[1]. However, unless specific behavior is created which predisposes the motion of the agents to a given area of the arena, we may expect that the time-averaged density[2] of robots in noncluster space is essentially constant. This means that on average, the robots themselves are not particularly interesting as their motion could very well be described by a natural gas-like condition. We expect this to be true no matter the configuration of pucks, provided that the configuration of pucks did not restrict accessibility of the robots to areas of the arena[3]. Thus, we concentrate on the motions of the pucks rather than that of the robots. This view relegates the robots to the role of part of the environment rather than an active part of the system. We are then interested in the time-averaged evolution of the system of pucks under the influence of the robot-enhanced environment.

Two assumptions are made.

1. The average density of robots in the arena is constant.

[1] The assumption that the arena is closed, though tacit, is realistic. The need to work in a physically bounded system is more pragmatic than anything else, and is a constraint of both artificial and natural multiagent systems.

[2] *Time-averaged density* of robots refers to the average number of robots carrying pucks in a given area of the arena over a long period of time with respect to the mobility of the robots. This can be calculated by observing a small area of the arena, and continually recording the number of robots in that area. The average number of robots in that area over the time interval approaches a constant number as time goes to infinity irrespective of the placement of the area in the arena.

[3] The assumption that the pucks do not restrict access to parts of the arena can be false. However, the effect of pucks reducing accessibility could be viewed as a something which tends to reduce the size of the arena, and inflate the size of the cluster.

2. Aside from the evaporation (and condensation) of pucks from (onto) a cluster, the behavior of pucks under the action of the robots will become, on average, equally dispersed throughout the space.

We consider the ability of the system to cluster under these assumptions.

Let's switch gears for a moment. We look at a system out of equilibrium in which there are a larger number of robots in one part of the arena carrying pucks than in any other area. In this case, we have a space in which there exists a gradient of puck density. In the simplest of cases, this gradient is linear in some direction $\widehat{\mathbf{x}}$. If we denote the distance along this axis by x, then we would assume that the density δ is a function of x, $\delta(x)$. We may assume that δ is monotonically increasing from one end of the arena to the other. It is interesting to understand what might happen here.

Recall that, all other things remaining equal, it may be assumed that the natural tendency is that the average density of moving pucks will become constant. What this means is that there will be a net flow of pucks from high density areas to low density areas. This will tend to happen irrespective of any other considerations. However, this means that in order to sustain the density gradient, there must be a puck sink at the low density area of the arena. This means that there may only be a sustained gradient if there is a net flow of pucks from the high density area to the low density area.

The opposite question is also important: what is the requirement for a net flow of pucks from one area of the arena to the other? If we do not let the robots exchange pucks, this means that a single robot carrying a puck must move from one region and drop it off in the other region. If the total density of robots is constant, this means that the number of empty robots on the low density side will be larger than that on the high density side. The end result is that a density gradient will occur and be maintained. This is an important consequence.

A net flow of pucks from one area of an arena to another may occur if and only if there exists a monotonic gradient in the density of robots carrying pucks between the areas in question, in which the density of robots carrying pucks is smaller at the end receiving the pucks.

In general, the rate of change of the size of a particular cluster, then, depends on two things. The first is the number of carriers interacting with the cluster. The second is the nature of the interaction of a carrier with the cluster in question. The rate of change of the size of a single cluster may be written as

$$\frac{dn_1}{dt} = -\delta_e F(n_1) + \delta_f H(n_1) \tag{31}$$

where the local density of empty robots δ_e and of robots carrying pucks δ_f are functions of position and time. In this equation, the functions F and H are analogous to previous functions, but now include the geometric properties of the arena and cluster position(s).

If we have two clusters, then the system is described by

$$\frac{dn_1}{dt} = -\delta_e(x_1) F(n_1) + \delta_f(x_1) H(n_1) \tag{32}$$

$$\frac{dn_2}{dt} = -\delta_e(x_2) F(n_2) + \delta_f(x_2) H(n_2) \tag{33}$$

If again, we assume that the number of pucks in the medium is stationary then we have that

$$\frac{dn_1}{dt} = -\frac{dn_2}{dt} \tag{34}$$

or

$$-\delta_e(x_1)\,F(n_1)+\delta_f(x_1)\,H(n_1)=\delta_e(x_2)\,F(n_2)-\delta_f(x_2)\,H(n_2) \tag{35}$$

In general, the density of carriers is not constant throughout the region. However, if we consider only the density of robots in the arena, either empty or filled, if follows that this density is, on average, constant. If we let D denote the time-averaged density of robots in the arena, and δ_e and δ_f denote the time averaged local densities of empty and full robots, respectively, then

$$D=\delta_e+\delta_f \tag{36}$$

We assume that we are working in a state of dynamic equilibrium, in which the local density of empty carriers and of full carriers is approximately constant. This means that even though there may be a net flow of pucks between clusters in the system we now consider, we assume that the local densities of the carriers carrying pucks and of empty carriers are remaining constant.

The way that the clusters evolve depends on the equilibrium states. In general, the system will behave in such a way as to move toward the equilibrium states of the individual clusters. The interaction between clusters will occur through the robot media, and there will be two competing tendencies. The first tendency will be to have time averaged densities of robots become equal throughout the space. The second will be the tendency of robots nearby clusters to remove or put down pucks, moving the local densities toward the equilibrium state of the cluster. We will derive the following equilibrium densities in Section 3.2, as it is a significant aside from our current discussion. For now, we state without proof that the equilibrium relations for a single cluster are

$$\delta_f=D\frac{1}{1+\frac{H}{F}}\text{ and }\delta_e=D\frac{1}{1+\frac{F}{H}}. \tag{37}$$

Then, if we denote the local densities around cluster one by δ_{e_1} and δ_{f_1}, with analogous expressions for cluster two, substituting equation (36) into equation (35),

$$-\left(D-\delta_{f_1}\right)F(n_1)+\delta_{f_1}H(n_1)=\left(D-\delta_{f_2}\right)F(n_2)-\delta_{f_2}H(n_2) \tag{38}$$

This can be rearranged to read

$$\delta_{f_1}=\frac{D\left(F(n_2)+F(n_1)\right)-\delta_{f_2}\left(F(n_2)+H(n_2)\right)}{\left(F(n_1)+H(n_1)\right)}\ . \tag{39}$$

This will be smaller than δ_{f_2} iff

$$D\left(F(n_2)+F(n_1)\right)<\delta_{f_2}\left(F(n_2)+H(n_2)\right)+\delta_{f_2}\left(F(n_1)+H(n_1)\right)\ . \tag{40}$$

$$\frac{\left(F(n_1)+H(n_1)\right)+\left(F(n_2)+H(n_2)\right)}{\left(F(n_2)+F(n_1)\right)}>\frac{D}{\delta_{f_2}}=\frac{F(n_2)+H(n_2)}{F(n_2)} \tag{41}$$

giving

$$F(n_2)\,H(n_1)>F(n_1)\,H(n_2) \tag{42}$$

or

$$\frac{F(n_2)}{H(n_2)}>\frac{F(n_1)}{H(n_1)} \tag{43}$$

This is the same condition found above for the case of the perfectly mixing robots. That is, if the ratio $G=\frac{F}{H}$ is monotonically decreasing, then the equilibrium density around larger clusters will be smaller than that around smaller clusters. δ_{f_1} being smaller than δ_{f_2} would seem to indicate that there are fewer robots carrying pucks nearby cluster one than nearby cluster two. This indicates that the robots are more successful in removing pucks from cluster two than from cluster one, and more successful in depositing pucks on cluster one than on cluster two. In this case, cluster one will tend to grow while cluster two tends to decrease. Moreover, there will be a net flow of pucks from cluster two to one. Put another way, this indicates that under the action of the robots, the smaller clusters will pump pucks into the media trying to reach their equilibrium, while the larger clusters will absorb these pucks, trying to reach their smaller puck or full carrier equilibrium.

Note that the relative density of empty or full carriers is a function of the design of the robot and the number of pucks in the cluster, as well as the cluster's detailed characteristics (eg. its geometry). Thus, as long as the design of a robotic system tends to create lower equilibrium densities of pucks around larger clusters, the larger of two clusters will tend to monotonically increase in size on average.

Perhaps most importantly, this result is easily generalizable to the multiple cluster regime. The minimum condition required is that the local density of robots carrying pucks induced by the cluster is a monotonically decreasing function of cluster size. Thus, nearby any given small cluster, the density will increase over that closer to larger clusters. Moreover, the smallest cluster will have the highest equilibrium density, and therefore can be seen to monotonically decrease in size, as in the previous section.

Mathematically, this can be shown as follows. Suppose that cluster q is the smallest cluster in the system, and that there are n clusters. Then we have

$$F(n_q)\,H(n_i) > F(n_i)\,H(n_q) \tag{44}$$

for all $i \neq q$. Then, we have

$$\sum_{i\neq q} F(n_q)\,H(n_i) > \sum_{i\neq q} F(n_i)\,H(n_q) \tag{45}$$

or that

$$\frac{F(n_q)}{H(n_q)} > \frac{\sum_{i\neq q} F(n_i)}{\sum_{i\neq q} H(n_i)} \tag{46}$$

which indicates that the effective cluster built from the detailed design of the other clusters tends to absorb pucks from the smallest cluster, mirroring our previous investigation of this topic.

Of course, the *efficiency* is another matter. This result does not help us to understand whether or not the system will quickly approach the desired equilibrium state. We return to this point in Section 4.

3.2 Induced densities

In (37), the equilibrium densities of empty and full robots around a cluster of a given size and interaction dynamic were stated without proof. This subsection is devoted to deriving these relations.

We assume that we have a single cluster located in a container of some kind. The robots interact with the cluster according to the previously-defined functions F and H, and may carry at most one puck at a time. We assume that there is no sink or source either of robots or of pucks.

In order for a cluster to be in equilibrium with a robotic (or other) medium, it must be that

$$\delta_e F = \delta_f H \tag{47}$$

where H and F represent the likelihood of a deposit of a puck to and the likelihood for the removal of a puck from a cluster of size n. If $D = \delta_e + \delta_f$[4] then

$$\delta_e F = (D - \delta_e)\,H\ . \tag{48}$$

[4]This condition implicitly requires the arena in which the robots are behaving to be closed or otherwise bounded.

Thus

$$\delta_e = D\frac{H}{F+H}, \ \delta_f = D\frac{F}{F+H} \tag{49}$$

These results may be rewritten as

$$\delta_f = D\frac{1}{1+\frac{H}{F}}, \text{ and } \delta_e = D\frac{1}{1+\frac{F}{H}}. \tag{50}$$

In the event that $\frac{F}{H}$ is a monotonically decreasing function of n, the evolution of pucks from a given cluster will decrease as its size increases. The equilibrium density surrounding a large cluster will be smaller than that around a small cluster. Thus, the pucks will collect into a single cluster as long as this condition is met. Happily, this is identical to our previous result, though it now contains no restrictions.

4 Efficiency

So far, we have investigated the question of whether or not a single cluster will form at all. This is compelling work to be sure, but not completely satisfying to the engineer. The main question to an engineer, after verification that the algorithm will work eventually, is how to increase the speed of an algorithm. In this Section, we discuss methods of providing algorithms that are at once capable of carrying out the task and of carrying it out efficiently.

4.1 Positive induced transport

In general the net transport between two clusters of differing sizes is an increasing function of the difference between their sizes. That is, if the difference between the sizes of two clusters is z then we expect the rate of transfer of pucks from one cluster to the other will be an increasing function of z. Mathematically, this may be expressed as

$$\frac{\partial \Theta}{\partial z} \geq 0 \tag{51}$$

where Θ represents the puck flux around the smaller cluster.

Suppose that we have two G functions, say G_A and G_B corresponding to two different interaction schemes of simple puck collecting robots. Then around cluster 1, the scheme A will have an equilibrium density of empty carriers given by

$$\delta_e^A(N_1) = D\frac{H^A(N_1)}{F^A(N_1)+H^A(N_1)} = \frac{D}{G^A(N_1)+1}, \ \delta_f(N_1) = D\frac{F^A(N_1)}{F^A(N_1)+H^A(N_1)} = D\frac{G^A(N_1)}{G^A(N_1)+1} \tag{52}$$

while near cluster two, the equilibrium density will be given by

$$\delta_e^A(N_2) = D\frac{H^A(N_2)}{F^A(N_2)+H^A(N_2)} = \frac{D}{G^A(N_2)+1}, \ \delta_f(N_2) = D\frac{F^A(N_2)}{F^A(N_2)+H^A(N_2)} = D\frac{G^A(N_2)}{G^A(N_2)+1} \tag{53}$$

Thus, we find that

$$\Delta\delta_e^A = \frac{D\Delta G^A}{(G^A(N_1)+1)(G^A(N_2)+1)} \tag{54}$$

and

$$\Delta\delta_f^A = -\frac{D\Delta G^A}{(G^A(N_1)+1)(G^A(N_2)+1)} \tag{55}$$

where $\Delta G = G(N_2) - G(N_1)$. Thus, if the transport of one algorithm characterized by G_A is greater than that of another characterized by G_B, we must have that

$$\frac{D\Delta G_A}{(G_A(N_1)+1)(G_A(N_2)+1)} > \frac{D\Delta G_B}{(G_B(N_1)+1)(G_B(N_2)+1)} \tag{56}$$

or that

$$\Delta G_A > \Delta G_B \frac{(G_A(N_1)+1)(G_A(N_2)+1)}{(G_B(N_1)+1)(G_B(N_2)+1)} \tag{57}$$

Of course, this is dependant on both strategies using the same technique for moving pucks around the arena. Any change in basic strategy would change the formalism above.

Suppose first that

$$G_B = G_A + r(N)$$

We wish to find the minimal condition on the function r which creates a second function G_B that is less efficient than G_A. Now, we may assume without a loss of generality that $r(N_1) = 0$. This gives us

$$\frac{\Delta G_A}{(\Delta G_A - r(N_2))} > \frac{(G_A(N_2)+1)}{(G_A(N_2)+r(N_2)+1)} \tag{58}$$

Since $\Delta G_A > r(N_2)$, we have that

$$\frac{(\Delta G_A - r(N_2))}{\Delta G_A} < \frac{(G_A(N_2)+r(N_2)+1)}{(G_A(N_2)+1)} \tag{59}$$

$$1 - \frac{r(N_2)}{\Delta G_A} < 1 + \frac{r(N_2)}{(G_A(N_2)+1)} \tag{60}$$

$$-\frac{r(N_2)}{\Delta G_A} < \frac{r(N_2)}{(G_A(N_2)+1)} \tag{61}$$

which leads us to

$$r(N_2)\left(\frac{1}{\Delta G_A} + \frac{1}{(G_A(N_2)+1)}\right) > 0 \tag{62}$$

or more simply that

$$r(N_2) > 0 \tag{63}$$

Thus, if we have two functions G_A and G_B, the latter will be less efficient if we simply have

$$G_B > G_A \tag{64}$$

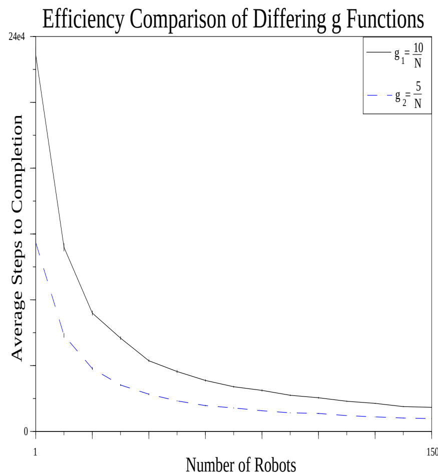


Figure 4.1.1: This figure illustrates the performance of the puck aggregation for a two cluster system with a number of different robot swarm sizes.

Figure 4.1.1 illustrates the performance of a several swarms of robots on a swarm aggregation task in which the g functions were given by

$$g_1 = \frac{10}{N} \tag{65}$$

and

$$g_2 = \frac{5}{N} \ . \tag{66}$$

Each performance measure for each swarm size was calculated using the aphysical model of clustering described in Section 2. 10000 pucks were initialized in equal proportions in two clusters, requiring the system to randomly fluctuate out of equilibrium and into clustering behavior. As predicted above, the clustering time of the swarm with a smaller g is significantly smaller than that of the swarm with the larger g, on average.

4.2 Induced transport

Previously, we assumed that the induced transport was a positive increasing function of the difference in densities between two clusters. The analog for perfect mixing is that the induced transport is a positive increasing function of the difference in the number of pucks in the cluster. In this subsection, we verify this assumption in the imperfect mixing case. We show this in this section, completing our analysis.

Previously, we found that for a given cluster of n_1 pucks the rate of growth of the cluster is

$$\frac{dn_1}{dt} = -\delta_e\left(x_1\right) F\left(n_1\right) + \delta_f\left(x_1\right) H\left(n_1\right) \tag{67}$$

Recalling that in equilibrium

$$\delta_e = T\frac{G}{G+1}, \ \delta_f = T\frac{1}{G+1} \tag{68}$$

We wish to verify that the rate of transport of pucks is an increasing function of the difference of induced densities between clusters. First, we assume that we have two clusters of differing size, say of sizes n_1 and of size n_2, respectively. These induce densities (in equilibrium) of

$$\delta_{1,f} = \frac{T}{G_1 + 1} \text{ and } \delta_{2,f} = \frac{T}{G_2 + 1} \tag{69}$$

Now, we assume that the flow of pucks is linearly dependent on the rate of change of the density of pucks. That is,

$$\frac{1}{\eta} F = \frac{\partial \delta_f}{\partial x} + \frac{\partial \delta_f}{\partial y} = -\frac{1}{\eta} \frac{dn_1}{dt} \tag{70}$$

where η is some proportionality constant associated with Einstein diffusion. If the cluster is much larger than the number of robots, this is approximately constant flow. This gives

$$F = \eta \left(\frac{\partial \delta_f}{\partial x} + \frac{\partial \delta_f}{\partial y} \right) = C \tag{71}$$

The net flow from any area is a constant C, assuming that no pucks are allowed to collect in any area of the arena. Solutions to this differential equation are

$$\delta_f = kx + ly + C_1 \tag{72}$$

Assuming that both clusters are on the $x - axis$, we may deduce that

$$k = \frac{\delta_{2,f} - \delta_{1,f}}{\Delta x} \tag{73}$$

Moreover, if the clusters are located within a bounded arena, we may conclude that $l = 0$, or that there is no gradient along the direction perpendicular to the radii connecting the clusters. Thus, we have

$$\delta_f \left(x \right) = \frac{\delta_{2,f} - \delta_{1,f}}{\Delta x} x + \delta_{2,f} \tag{74}$$

giving

$$F = \eta \left(\frac{\delta_{2,f} - \delta_{1,f}}{\Delta x} \right) = C \tag{75}$$

Clearly, if

$$\frac{\delta^1_{2,f} - \delta^1_{1,f}}{\Delta x} > \frac{\delta^2_{2,f} - \delta^2_{1,f}}{\Delta x} \tag{76}$$

where $\delta^1_{1,f}$ represents the induced density at cluster one for scheme one, $\delta^2_{1,f}$ represents the same for control scheme 2, and so on. Thus, as long as this difference is larger, for scheme two, then the induced flow is larger. The rest of our conclusions from above follow quite nicely.

5 Clustering systems

In this Section, we investigate several forms of g. We illustrate that these forms have the predicted behaviors in both the pair of cluster cases and the multiple cluster cases. We provide experimental evidence to support our approximation of the amount of variability in the number of pucks held by robots.

5.1 Simple example

In this section, we examine the behavior of a aphysical puck-agent system. The system is that described in Section 2 in which the agents randomly choose clusters from which to take pucks or to which to deposit pucks. The cluster in question is altered as described in Section 2. We choose a particularly simple form for f and h in order to illustrate the long term behavior of the system in three different regimes. These regimes of behavior are characterized by a monotonically decreasing g, a constant g, and a monotonically increasing g.

Assume that f and h have the form

$$f(n) = cn^{\alpha} \tag{77}$$

$$h(n) = cn^{\beta} \tag{78}$$

where c, α, and β are some constants. Then,

$$g(n) = n^{\alpha-\beta}. \tag{79}$$

Thus the cluster growth condition holds if $\beta > \alpha$, in which case g is monotonically decreasing. When $\beta = \alpha$, g is constant . If $\beta < \alpha$, g is monotonically increasing.

Figures 5.1.1-5.1.3 present the results of simulations done in the non-embodied model on a number of clusters initialized with nonzero sizes. Each of these simulations employs 50 robots, and the model is as given above, with various values for α and β. Each one runs to completion if clustering occurs or for a small number of steps, allowing the outcome to be clearly ascertained.

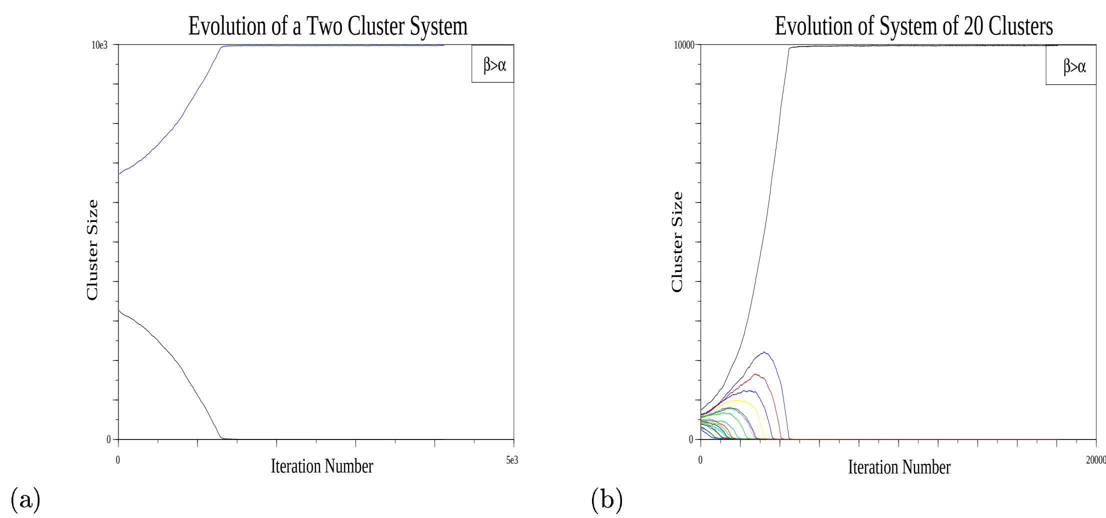
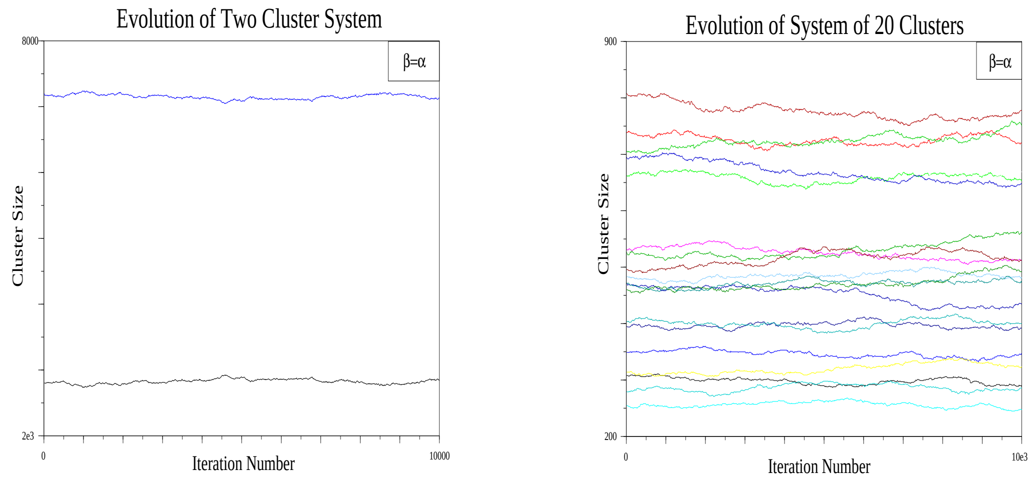
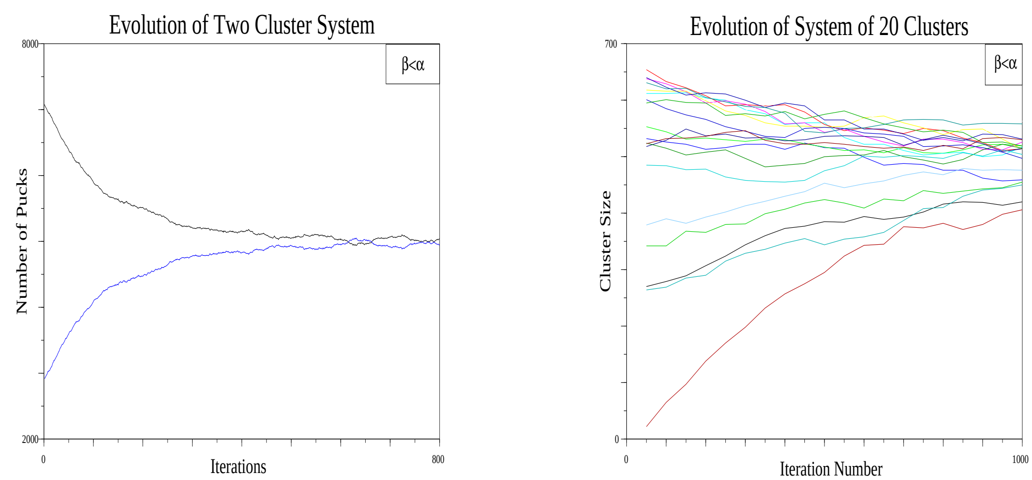


Figure 5.1.1: This illustrates the behavior of a system when $\alpha = 1.0$ and $\beta = 2.0$. $c = 0.00001$ for two clusters, and $c = 0.0001$ for twenty clusters. As expected, both systems converge to a system with one cluster containing all the pucks.



(a) (b)

Figure 5.1.2: This illustrates the behavior of a system when $\alpha = 1.0$ and $\beta = 1.0$. $c = 0.00001$ for two clusters, and $c = 0.0001$ for twenty clusters. As expected, neither system converges.



(a) (b)

Figure 5.1.3: This illustrates the behavior of a system when $\alpha = 1.1$, and $\beta = 1.0$. $c = 0.00001$ for two clusters, and $c = 0.0001$ for twenty clusters. As expected, neither system converges.

Figures 5.1.1 through 5.1.3 illustrate the convergence states of the system under the action of the robot swarm. The results are as expected. Only the case in which $\beta > \alpha$ produces a stable single cluster. Interestingly in Figures 5.1.1, 5.1.2, and 5.1.3, the clusters seem to have a significant amount of noise in their final sizes. This is the topic of an upcoming paper describing ways to reduce this noise [13].

5.2 Physical robot model

The long term behavior of physical robotic systems are difficult to predict in general because of the tight coupling between the emergent properties of the system in which the robot behaves and the behaviors of the robots. Many behaviors can be generated which produce structures that are interesting and which later influence the behaviors, in turn changing the structure. This can be extremely difficult to predict. There is as of yet no general theory which indicates how a complex dynamic system will evolve. This would seem to be a serious problem when attempting to develop a theory of swarm-based behavior.

Rather than attempt to solve this problem, we focus on a few behaviors that would seem to avoid this problem by design. These behaviors may be characterized as rotationally symmetric in that their behavior when approaching a circular cluster from any given side is identical. Moreover, these behaviors would seem to be designed to restore a cluster to approximately circular shape when the shape deviates from circular. Because of these properties, more completely addressed elsewhere, we may view the clusters as approximately circular in the following analyses.

In many different robotic systems puck deposit or removal are decided based on the same criterion. If the criterion is met then the robot will remove a puck. Otherwise, it will not. Moreover, if it is not, the robot will deposit a puck if it is carrying one. In this case, we may write $f = \sigma\rho$ and $h = \sigma(1-\rho)$ and

$$g = \frac{1-\rho}{\rho} = \frac{1}{\rho} - 1. \tag{14}$$

Recall that g must be a monotonically decreasing function of the number of pucks in order for the clusters to converge to a single cluster. This means that ρ must be a monotonically increasing function of the number of pucks in the cluster in order for the system to evolve to a single cluster state. Note that since g cannot be negative, ρ must be smaller than 1. Moreover, as g is a unitless number, so too must ρ be.

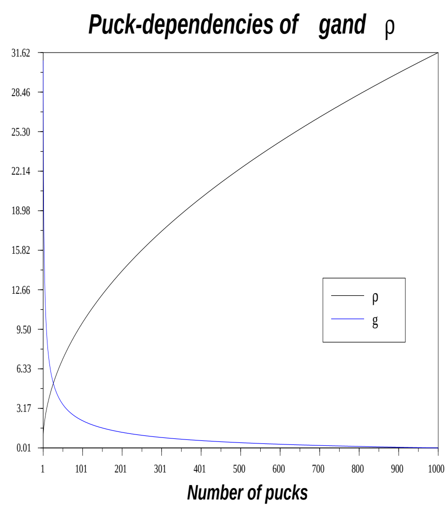


Figure 5.2.1: This figure gives the behavior of the function g with increasing ρ. The monotonic decrease is all that is required for g to lead to single clustering-behavior of the system.

For real robotic systems, the jump from physical design to functional form is somewhat difficult to carry out. However, let us suppose that we use a robot which carries a simple passive visual system on board. If we suppose that the robot is acting within an arena whose size allows accurate discrimination of two different clusters along the same line of sight, we may assume that the camera system can be used to accurately determine the depth of a given cluster. If then, upon contact, the cluster fills a sufficient region of the visual field, the cluster may be classified as large. If however, the cluster does not fill a sufficient region of the visual field, it may be classified as small. The situation is illustrated in Figure 5.2.2.

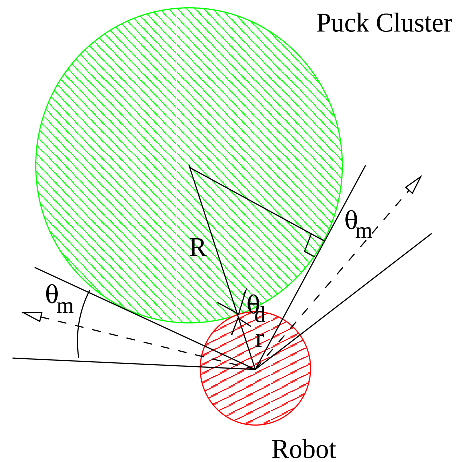


Figure 5.2.2: This figure illustrates the robot's interaction with the cluster. The robot will determine the density characterization for clusters whose pucks fall within the angle of interaction. All other interactions will be viewed as obstacle avoidance.

We assume that the visual field of the robot subtends an angle denoted by $2\theta_m$. In this model a robot will only deposit a puck if the direction of the robot is such that the cone falls completely within the cluster and it is currently carrying a puck. If the cone only falls partly within the cluster then the robot, if it is able, will remove a puck. It follows therefore that the range of approach angles for removal and deposit sum to the total angle of interaction of the robots cone $2\theta_m + 2\theta_d = \Theta$, where θ_m is the interaction angle and $\theta_d = \arcsin\left(\frac{R}{R+r}\right) - \theta_m$ is the deposit angle.

Using this construction gives

$$h = \sigma(R)\rho(R,r) \tag{80}$$

and

$$f = \sigma(R)(1 - \rho(R,r)) \tag{81}$$

where σ is some cross section based on the radius of the cluster and

$$\rho = \begin{cases} 0 \; if \; b < 0 \\ b \; if \; b \geq 0 \end{cases} \tag{82}$$

$$b = \frac{2\arcsin\left(\frac{R}{R+r}\right) - \theta_m}{2\arcsin\left(\frac{R}{R+r}\right) + \theta_m} = 1 - \frac{2\theta_m}{2\arcsin\left(\frac{R}{R+r}\right) + \theta_m}. \tag{83}$$

b and ρ both increase with R which, in turn, increases with the number of pucks n. Thus g for this model is monotonically decreasing and the system will form a single cluster.

Figure 5.2.3 illustrates a typical system evolution for a simulated system of this design. Initially, we have two clusters with nearly equal sizes. After many iterations, the migration of pucks from one cluster to the other causes the smaller of the two clusters to completely evaporate and the larger of the two clusters to absorb nearly all of the liberated pucks.

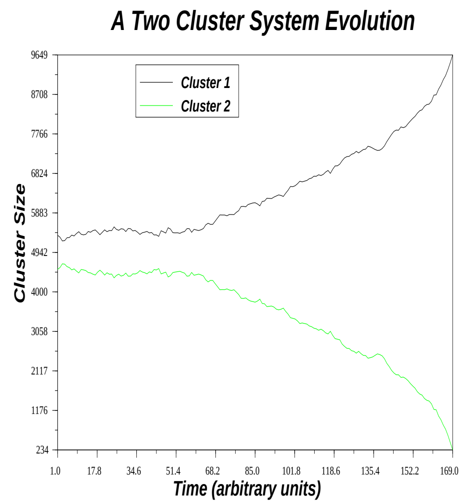


Figure 5.2.3: This figure gives the behavior of a two-cluster system under the influence of the robotic system described in this section.

5.3 Robot interactions

In Sections 2 and 3, we assumed that the robot swarm would not absorb or evolve pucks on a large scale; the derivative of the number of pucks contained in the swarm could be ignored. We now turn to this assumption, offering empirical evidence that supports it. We present data collected in the simulations given in Section 4.1. In these simulations, 50 robots make up the swarm.

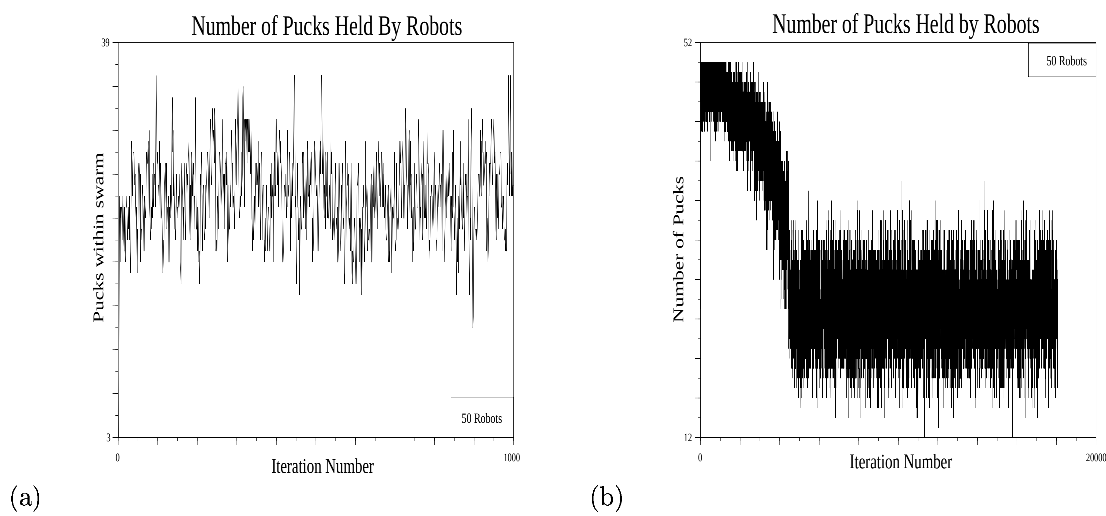


Figure 5.3.1: These figures give the robot swarm puck occupancy. The swarm is at full capacity when the number of pucks in the swarm is 50. However, this never becomes the case in the (a) constant clusters size case, and once a single cluster is formed, this ceases to be the case (b) in the converging cluster case.

Figures 5.3.1 gives the evolution of systems of clusters in differing swarm conditions. In (a) the swarm does not change the cluster sizes appreciably, and so the occupancy of robots remains constant throughout the simulation. In (b), the swarm does converge, and the occupancy transitions from a full occupancy to a medium occupancy when the cluster is finally formed. This final occupancy is the equilibrium occupancy value which we return to in Section 6.

6 Predictions of the model

6.1 The robots of Beckers et. al.

Several researchers have reported construction of robots that carry out puck clustering behaviors. One such set of robots, detailed in the work of Beckers et. al. and Maris and Boekhorst is capable of clustering pucks by pushing them around a bounded arena. Each robot is equipped with a "gripper" which pushes the pucks. This gripper is designed to detect when a large number of pucks are being pushed, and to trigger a backing up behavior when encountering too many pucks, thereby leaving pucks it is currently pushing where they are. The robot has four possible interactions with the pucks.

1. Pick up a single puck
2. Pick up two pucks
3. Drop a single puck
4. Drop two pucks.

Each robot will travel in straight lines if not encountering pucks in the path until reaching the boundary of the arena. When it reaches the boundary, the robot will execute a random turn and continue traveling in a straight line. If it encounters a cluster, it will continue on providing that it encounters sufficiently few pucks. This will occur if the robot simply hits the edge of the cluster, as in Figure 6.1.1, or if the cluster itself is made up of sufficiently few pucks. If we assume that the width of a puck is a, then

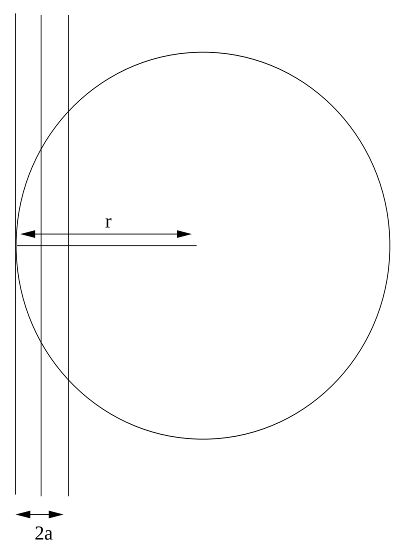


Figure 6.1.1: The modes of interaction of the robots and clusters in Beckers et. al. clustering paper.

the robot has a probability of $\frac{a}{r}$ of pulling off a single puck, $\frac{3a}{2r}$ of pulling off two pucks, assuming the robot is not already pushing pucks and that pucks tend to arrange themselves hexagonally in the cluster, as has been reported [11]. Likewise, it has a probability of dropping off a single puck if pushing a single puck of $\frac{2r-3a}{2r}$, and of 1 if it is already pushing two pucks. Thus, we may write an expression for h and f as

$$h = \lambda \frac{(2r - 3a)}{2r} n_1 + 2\lambda n_2 \tag{84}$$

and

$$f = \lambda \frac{a}{r} n_1 + \lambda \frac{a}{r} n_0 + 2\lambda \frac{3a}{2r} n_0 \tag{85}$$

where n_0, n_1, and n_2 represent the number of robots carrying 0, 1, and 2 pucks, respectively, and λ is a proportionality constant indicating the likelihood of interacting with the cluster. Then

$$g = \frac{a}{r} \frac{2n_1 + 8n_0}{\left(2 - 3\frac{a}{r}\right) n_1 + 4n_2} \tag{86}$$

Now, if $r \gg a$ then

$$g \simeq \frac{a}{r} \frac{(n_1 + 4n_0)}{n_1 + 2n_2} \tag{87}$$

This will generally be decreasing if n_0, n_1, and n_2 are constant or decreasing as the sizes of the clusters increases. While this has been established earlier in this paper for aphysical agents, we may only assert that this is a likely general behavior for real systems. Further investigation would be required to establish this fact.

6.2 The robot of Martinoli, Ijspeert, and Mondada

Martinoli, Ijspeert, and Mondada [10]report the use of a clustering experiment in which a khepera robot manipulates 'seeds' and attempts to place them in long semi-linear "clusters".

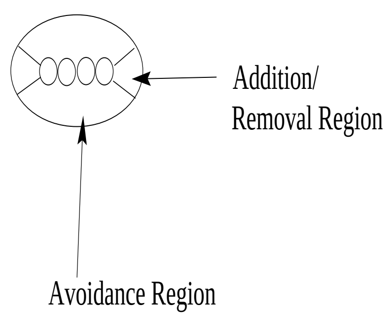


Figure 6.2.1: This gives the addition and subtraction regions of interaction of a khepera robot approaching a cluster of "seeds".

These clusters have two ways of being modified. These are having a seed added to the end of the cluster, or having a seed removed from the cluster. We plot the ratio of probabilities that these will occur reported in [10] in Figure 6.2.1.

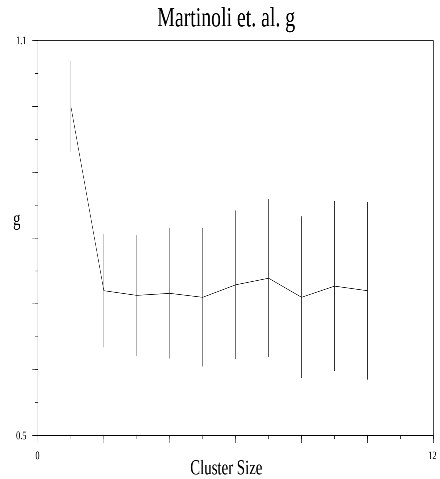


Figure 6.2.2: g as reported in [10]. The error is a result of estimations of the numerical values for h and f at each point.

This is clearly not monotonically decreasing, and so therefore *does not* satisfy our criterion for *monotonic clustering*. Indeed, this is precisely what is reported in [10], as the authors note that the behavior is somewhat driven by a probability of clusters to disappear, after which time an irreversible system change has occurred. The performance of the system is expected to require long times to converge. The time required to remove the larger clusters and add their constituent parts to the smaller clusters would seem to be an exponential function of the length of the cluster. Indeed, this is what is observed. We would characterize this behavior as *pseudoclustering*, in which no direct clustering behavior is responsible for the clustering itself, but it is rather the effect of irreversible random events which may be built upon by further irreversible random events of the same type. Moreover, we have seen clustering of this type before. In Section 5.1, we illustrated behavior of clusters using a constant g. The behavior of that system is identical to this.

We illustrate this behavior in Figure 6.2.3, generated from the aphysical model.

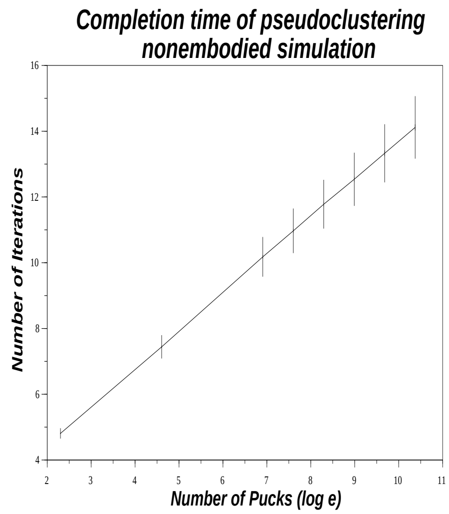


Figure 6.2.3: This figure illustrates the exponential dependence of convergence time of pseudoclustering events on the number of pucks in the system. Such behavior is the hallmark of pseudoclustering systems, which must then be excluded from practical scalable construction mechanisms.

Thus, using our formalism, we can predict the correct long-time outcomes of the two systems introduced in Section 2. Moreover, we should be able to predict other systems using similar formalism. This illustrates the predictive power of this design paradigm.

7 Cluster evolution under the influence of different forms of g.

If we recall that g represents the effective behavior of robots over a large time, we can imagine that this is the effect of the swarm, and thus we can step back from the system and ask questions about the design of the swarm around a specific goal. Let us examine this aspect of swarm engineering a bit more directly .

Swarm engineering [14], formally stated, requires that one creates a swarm criterion and generates behaviors which satisfy this criterion. Previously, we conditioned our swarm design on the desire to obtain a single cluster at the completion of the construction phase. This design requirement led us to the *local swarm criterion* that

> *In a two cluster system, the largest cluster should increase in size, while the smaller decreases in size. In a many cluster system, the smallest cluster should decrease in size.*

This criterion led us to the general condition that g should be strongly monotonically decreasing. We now turn to the generation of further behaviors.

7.1 Increasing g

At the end of Section 3.1, we found that the condition required for creation of a single cluster of pucks independent of the initial conditions of the system was that g was a strongly monotonically decreasing function of N. In this section, we investigate implications of the opposite condition on g. In the next Section, we give the effect of these behaviors on various forms of g given various initial states.

First, we reverse the condition given in equation (10). Then

$$g(n_2) = \frac{f(n_2)}{h(n_2)} < \frac{f(n_1)}{h(n_1)} = g(n_1) \ . \tag{88}$$

This expresses the reverse condition. It is straightforward that this is true iff

$$f(n_1)\,h(n_2) - f(n_2)\,h(n_1) > 0 \tag{89}$$

Since we have previously shown that

$$\frac{dn_1}{dt} = n_c \frac{h(n_1)\,f(n_2) - h(n_2)\,f(n_1)}{f(n_1) + f(n_2) + h(n_1) + h(n_2)} \tag{90}$$

this together with (89) gives us that

$$\frac{dn_1}{dt} < 0 \tag{91}$$

Moreover, since

$$\frac{dn_2}{dt} = n_c \frac{h(n_2)\,f(n_1) - h(n_1)\,f(n_2)}{f(n_1) + f(n_2) + h(n_1) + h(n_2)}, \tag{92}$$

we have

$$\frac{dn_2}{dt} > 0. \tag{93}$$

This can be summed in the following Theorem.

> **Theorem**: Suppose we have a two cluster system of pucks. If the ratio g is as defined above, the
> cluster with the larger g value loses pucks to the cluster with the smaller g value.

Note that this is also true for effective clusters in the same way as for real clusters, allowing this Theorem to be applicable to the multiple cluster system as well as the single cluster system. This means that any system in which one cluster will have a higher g value than the other will tend to lose pucks to the other, irrespective of the clusters' relative size. Now we turn to several rather specific peaked functions of g, in which the long time behavior is completely worked out.

7.2 Stability of multi points

The possibility of using non-monotonic forms of g creates a very interesting possibility. With nonmonotonic forms of g, we can, in certain circumstances, create clusters of *differing* sizes. Let us look at this by first defining what we will call multi points. *Multi points* are those points in phase space in which the system settles during which the system has two or more stable clusters in dynamic equilibrium with one another. A multi point refers to the size of the cluster, rather than the value of g.

We wish to understand under what conditions we can expect these multi points to be stable. In order for them to be stable, there must be a restoring force when the system moves out of equilibrium. Thus, let us start with the condition for the existence of a multi point. We must have two clusters of sizes N_1 and N_2, which are not necessarily equal. We must have that

$$g(N_1) = g(N_2) \tag{94}$$

Then if we change the size of cluster 1 by some amount Δ, by our conservation of pucks condition, we must change the size of cluster 2 by the amount $-\Delta$. Thus we have that $g(N_1)$ becomes $g(N_1+\Delta)$ and $g(N_2)$ becomes $g(N_2-\Delta)$. Now, in order for a restoring force to exist, cluster 1 must start loosing pucks. This can generally occur (Theorem from Section 7.1) if

$$g(N_1+\Delta) > g(N_2+\Delta)\ . \tag{95}$$

If Δ is small, or these are linear regions, then we may approximate $g(N_1+\Delta)$ as

$$g(N_1+\Delta) \approx g(N_1) + \Delta \left.\frac{\partial g}{\partial N}\right|_{N_1} \tag{96}$$

and

$$g(N_2-\Delta) \approx g(N_2) - \Delta \left.\frac{\partial g}{\partial N}\right|_{N_2}\ . \tag{97}$$

Thus, noting that $g(N_1) = g(N_2)$, the condition becomes

$$\left.\frac{\partial g}{\partial N}\right|_{N_1} > -\left.\frac{\partial g}{\partial N}\right|_{N_2} \tag{98}$$

This means that the two regions must have two conditions met. First, the region in which cluster 1 is located must be increasing. Second, the region which cluster 2 is located must not be decreasing faster than the region around cluster 1 is located. This puts a strong bound on the appearance of multi points.

7.3 Forms of g

Let us now consider the behavior of the system under different forms of g. Some of these forms of g should produce multi points, which would seem to be a first necessary step in development of construction techniques under the action of multiple robotic systems. Some of the forms of g should produce, as we've seen, single clusters, while others will produce multiple clusters, some sets of which contain clusters of differing sizes. We examine one of each of these cases, and summarize the other cases in a table.

7.3.1 Minimum with Equal Sloped Linear Regions

Let us suppose that we have a g made up of the juxtaposition of two linear regions which have the same magnitude of slope. Then we have three possible situations, with associated possible outcomes.

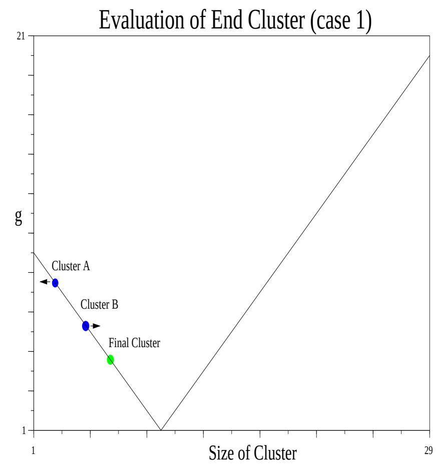


Figure 7.3.1: This gives the behavior of the pair of clusters when initiated on the decreasing region of g. The system evolves to one having a single cluster, as expected from above.

In Figure 7.3.1, we plot the evolution of a pair of clusters when initiated on the decreasing region of g. In this case, Cluster 1 will grow, absorbing Cluster 2, and eventually resulting in a single cluster of all the pucks. This is commensurate with the results of Section 3.

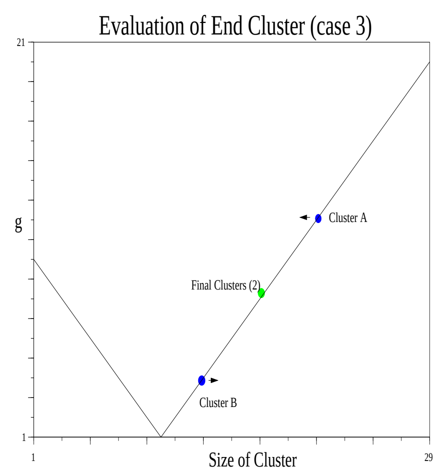


Figure 7.3.2: This gives the evolution of two clusters initiated on the increasing region of g. In this case, the clusters will evolve to the same size, equipartitioning the pucks.

If we instead put the clusters initially on the increasing region of g, the situation radically changes. Since the larger cluster now has the larger g value, it tends to lose pucks to the smaller cluster. Thus, the two clusters approach each other in size, and eventually become equal in size.

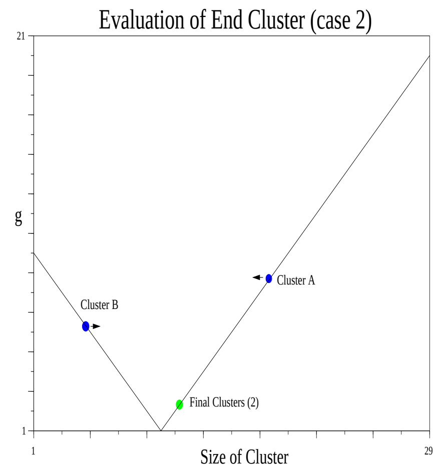


Figure 7.3.3: This figure illustrates the system evolution when the clusters are initialized on different sides of the minimum, with the larger cluster at a higher g value than the smaller.

Figure 7.3.3 illustrates the system evolution when the clusters are initialized on differing sides of the minimum with the larger cluster at a higher g value than the smaller cluster. In this case, the larger cluster will lose pucks to the smaller cluster. Since the slopes are equal, the loss of pucks is directly proportional to the relative loss of g. Thus, the larger cluster cannot "catch up" in g to the smaller. The result is that the process will continue until the initially smaller cluster passes the minimum of g and begins rising in g to meet the initially larger cluster. We end up with a system of two identically-sized clusters.

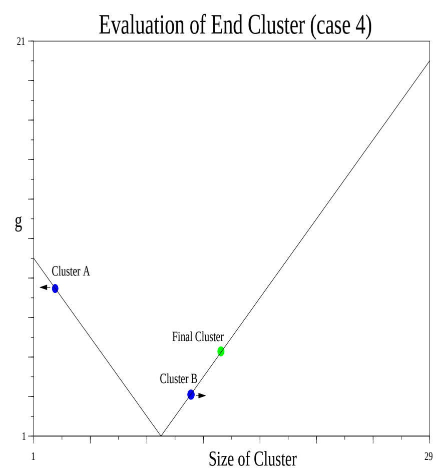


Figure 7.3.4: This Figure illustrates the evolution of a system of two clusters initially on differing sides of the minimum, with the smaller cluster at higher g.

In Figure 7.3.4, we illustrate the evolution of a system of two clusters initially on differing sides of the minimum. Initially, if the smaller cluster is at a higher g than the larger cluster, it will lose pucks to the larger cluster. This results in the gradual decrease in size of the smaller cluster until it is absorbed completely by the larger cluster.

Thus, if both regions have the same slope, there are two possible outcomes: the clusters merge to form one cluster, or the clusters equipartition the pucks.

7.3.2 Minimum with Large Region 2

In this SubSection, we assume that the magnitude of the slope of region 1 is smaller than that of region 2.

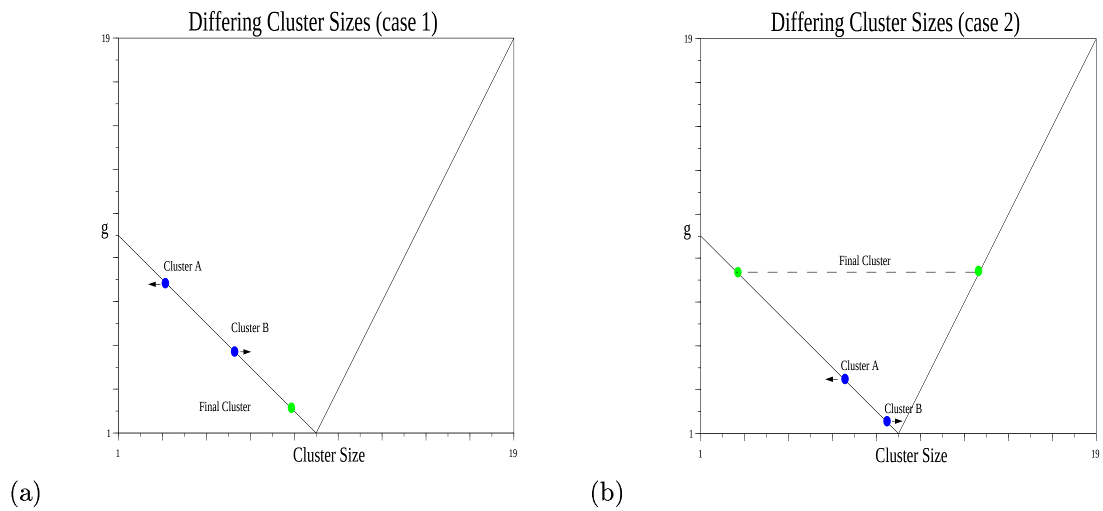


Figure 7.3.5: The evolution of a system of two clusters initialized on the increasing side of the g. This has the capability to evolve to a one cluster or two cluster system, with the two cluster system containing clusters of differing sizes.

Figure 7.3.5 illustrates the behavior of the system of two clusters both initialized on the decreasing side of g. In this case, the system has two possible stable outcomes. In the first, the pucks are completely absorbed by the larger cluster, yielding one final cluster (a). In the second, (b) the slope of the second region is sufficient for the g value of the larger cluster to "catch up" to that of the smaller cluster, yielding a stable configuration.

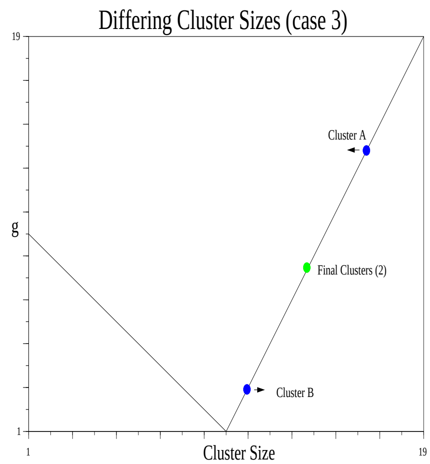
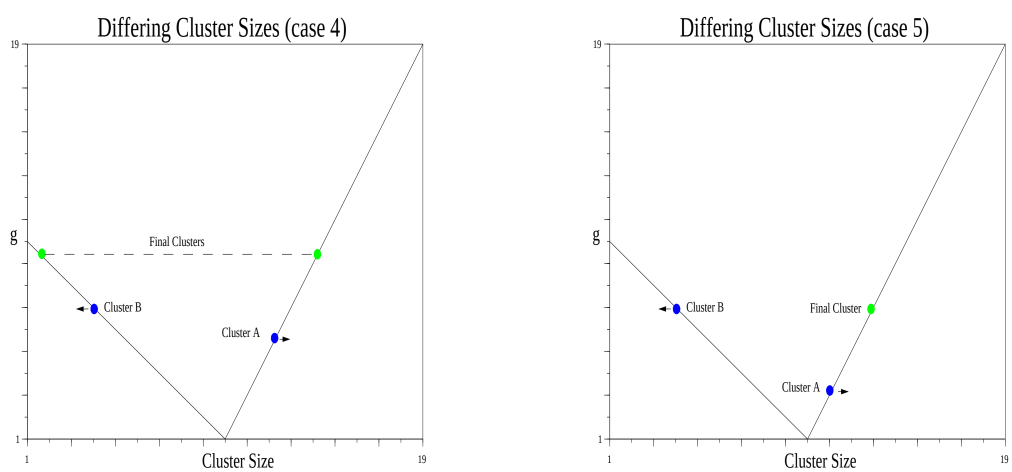


Figure 7.3.6: This gives the behavior of a two cluster system initialized on the increasing part of g. As before, the system forms a stable configuration of two clusters of the same size.

Figure 7.3.6 illustrates the behavior of a system of two clusters initially in the increasing region of g. As before, this system forms a stable configuration of two clusters with the same size.



(a) (b)

Figure 7.3.7: This Figure illustrates the system evolution of a system of two clusters initialized in opposite sides of the minimum in g. Two outcomes are possible : a system of two *different sized* clusters, or one cluster.

In Figure 7.3.7, we give the evolution behavior of the system of two clusters initially on different sides of the minimum in g in which the larger cluster has a smaller g value than the smaller. This produces two different results depending on the relative slopes and positions of the clusters. In the first behavior, the slope on the increasing side is large enough that the g value of the larger cluster "catches up" to that of the smaller. At this point 7.3.7(a) the system forms a stable configuration of different sized clusters. The 7.3.7(b) second behavior is observed if the slope of the second region is insufficient for the g value of the larger cluster to "catch up" to that of the smaller cluster. In this case, a single cluster containing all the pucks is formed.

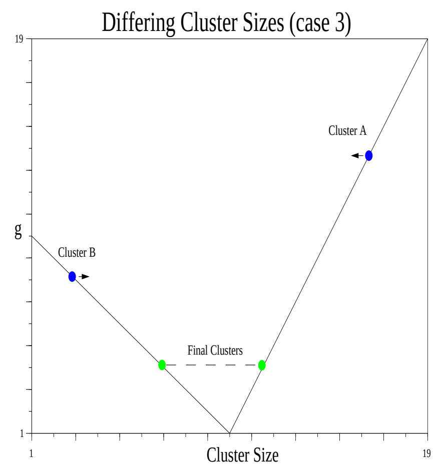


Figure 7.3.8: The evolution of a system of two clusters of differing sizes separated across the minimum in g in which the larger cluster has a larger initial g. The system evolves to a stable state containing two clusters of differing sizes.

In Figure 7.3.8, the system is initially comprised of two clusters of different sizes on opposite sides of the minimum in g. The larger cluster is at a higher g level than the smaller cluster, and evolves pucks which are absorbed by the smaller cluster. This process continues until the g levels become equal, which may happen before the clusters become the same size, as the slope of the second region is now greater than that of the first region. If the g levels do not become equal before the cluster sizes are equal, the system will produce two stable clusters of equal size.

Thus, there are now three possible outcomes if g is made up of two different regions in which the slope of the second has a larger magnitude than that of the first region. In some cases the system evolves to a stable state of one cluster, while in others the system evolves to a stable state containing two identical clusters. However, surprisingly, if the system is appropriately primed, it may evolve to a stable set of two differently sized clusters. This suprising result may be able to be exploited in order to create some interesting behavior.

7.3.3 Summary of All Cases

The main difference between the two cases given in Sections 7.3.1 and 7.3.2 lies in the differing slopes between the two regions of g. In Section 7.3.1 where the magnitude of the slopes of each region are equal, we find the formation of single versus multiple clusters to be dependent solely upon the initial size distribution of the clusters between the regions. Here the two clusters lose and gain pucks at equal rates, allowing either for convergence towards two clusters of equal size or the the disappearance of the smaller one entirely. In section 7.3.2, however, g consists of two linear regions with slopes of differing magnitude. Thus a cluster of a size on region two will either lose or gain pucks at a faster rate than a cluster whose size falls in region one. Such a nonhomogeneous g now provides us with a possible evolution towards a stable system of two clusters of differing sizes as well as single and multiple clusters of the same size, as the corresponding gs for each cluster may yet become equal. That is, the faster rate of change for one g drives the cluster towards equilibrium with the same g for a cluster of another size. This also occurs for other forms of composite g as well. We now summarize the results of other composite gs in the following table:

Region of g with Slope of Greater Magnitude	Initial Location of A	Initial Location of B	Cluster with Higher g	Number and Type of Clusters Formed
Neither	1	1	A	1
Neither	1	2	B	2 (same size)
Neither	1	2	A	1
Neither	2	2	B	2 (same size)
Region 1	1	1	A	1
Region 1	1	2	B	2 (same size)
Region 1	1	2	A	1
Region 1	2	2	B	2 (same size)
Region 2	1	1	A	1 / 2 (different size)
Region 2	1	2	B	2 (same or different sizes)
Region 2	1	2	A	1 / 2 (different size)
Region 2	2	2	B	2 (same size)

Figure 7.3.8: The evolution of a system of two clusters of differing sizes separated across the minimum in g in which the larger cluster has a larger initial g. The system evolves to a stable state containing two clusters of differing sizes. Note that this table assumes that cluster B is the larger cluster in all cases.

What is interesting is that there are a limited number of different cases in which it is possible to generate differently sized clusters. These regions are most interesting to the engineer after the development of construction techniques, as these regions allow the creation of several differently sized objects or constructs, which would seem to be a necessary first step towards construction. We demonstrate this more fully in Section 7.4.

7.4 Proper swarm engineering

An engineering tool is virtually useless if it cannot be used to create systems whose properties are well known in advance and which may be used to accomplish some goal. In this brief Section, we design an aphysical robot swarm which is capable of creating two clusters of predefined sizes. Initially, a particularly simple system to create is one of two clusters in which all the pucks are initially in one cluster and end up in a predefined proportion in a second cluster.

Suppose that we have a total of N pucks, and we wish to create a system of pucks which has N_1 pucks in the larger cluster and N_2 pucks in the smaller cluster. As initial conditions are easy to control, we assume that the initial distribution is an easily generated structure.

Now, we suppose that we have two linear regions l_1 and l_2 which meet at a minimum of g for the two clusters. We assume also that the robot has an accurate method of determining the size of the cluster, deferring investigation of limited ability. Without a loss of generality, we may assume that $g\left(N=1\right)$ while $g\left(N=0\right)=0.5$. If this is the case, and we wish to have the system settle at N_1 and $N_2=N-N_1$ then we must have

$$\frac{1-g_{eq}}{N_1}=m_2 \tag{99}$$

and

$$\frac{g_{eq}-0.5}{N1}=m_1 \tag{100}$$

These yield the linear equations

$$l_1 : y=\left(\frac{g_{eq}-0.5}{N_1}\right)x+0.5 \tag{101}$$

$$l_2 : y = \left(\frac{1 - g_{eq}}{N_1}\right) x + \frac{g_{eq}N - N_2}{N_1} \tag{102}$$

giving a minimum at

$$\frac{N(1 - g_{eq}) - 0.5N_1}{(1.5 - 2g_{eq})} = x \tag{103}$$

so that if $g_{eq} = 0.45$ then

$$\frac{1.1N - N_1}{1.2} = x \tag{104}$$

Note that now the linear equations are given by

$$l_1 : y = \left(\frac{-0.05}{N_1}\right) x + 0.5 \tag{105}$$

$$l_2 : y = \left(\frac{0.55}{N_1}\right) x + \frac{0.45N - N_2}{N_1} \tag{106}$$

Thus, we have a design criterion. If we let the $f = 1$ for all cases, which means that the robot will pick up every puck it encounters, and let the probability to drop be a function of size as prescribed in the above linear equations. We graph the evolution of a two-cluster system initialized with one cluster containing all the pucks and the second containing none.

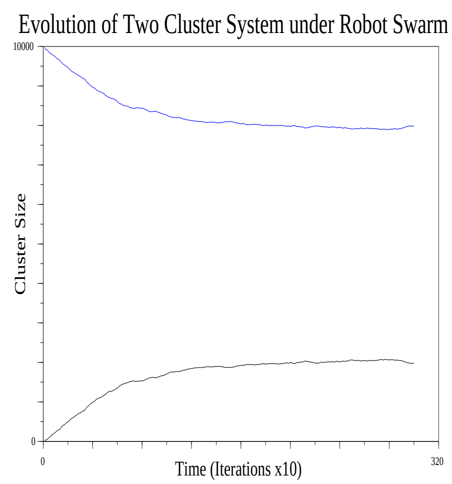


Figure 7.4.1: This figure illustrates the use of a swarm engineering technique designed to produce two clusters of a prespecified 4:1 ratio.

In this Figure, the sizes of the clusters are plotted as a function of time. Initially, all pucks are in the larger cluster. However, very quickly, the clusters begin exchanging pucks, and continue to have a net flow to the smaller cluster until the desired system is produced. Generalizations of this technique using a rough binary estimation of size, or a noisy size estimation produce similar results, with no significant degradation (up to 10% error). Most importantly, this can be easily generalized to multiple clusters, to clusters of differing size, and of a specific spatial design. These further possibilities require more investigation.

8 Discussion and concluding remarks

Groups of cooperating robots have the potential to carry out a wide variety of tasks which require little or no intelligence at the single agent level, but which allow a task which would seem to require intelligence to be completed. In the natural world, these systems have been shown to be capable of carrying out tasks which seem to be complex and despite the meager intelligence requirement. Indeed, they have been shown to be more than a simple sum of the parts, forming an entirely new whole.

The robustness and expanded capability of multiple robotic systems has been investigated by a number of groups. However, typical investigations consist of the creation of one or more working models, and little theory indicating the ability of the system to complete the task. Perhaps more importantly, typically, the creation of a class of possible strategies is ignored entirely. This would seem to be the result of the difficulty in creating working robotic systems in the first place, as this is a daunting task in itself.

In this work, however, we have turned the design paradigm on its head. What we've done is ask what the general conditions must be in order to design a system of puck collectors which will cluster pucks into a single cluster. Once this condition has been satisfied, any design paradigm can be shown to lead to the robust clustering of the system. This condition does not take into account the efficiency of the algorithm, but instead deals only with the final global design goal. Further work must be completed in order to attend to the question of efficiency.

What this does is allow a great generality and freedom in our design. We may now use exceptionally simple designs to complete the same global design goal. This allows us to build robots with extremely simple algorithms which will complete our task. What we have then achieved is a general methodology for creating extremely simple algorithms which yield a desired global behavior. Indeed, we have begun to unravel the idea of *emergent properties* and have replaced them with *global design properties* which may be seen to be different, in many cases, from those simple properties built into the robot. As an example, we may see that there is nothing in the simple robot which picks up pucks and shuttles them from group to group which would indicate that this would yield a global collection behavior. However, this is indeed the end result, despite the fact not being immediately obvious without the advantage of hindsight or deep insight.

Further work in this area may tackle the use of stigmergic communication as a minimal necessary means of completing tasks. In order for this to be required, however, it must be necessary to recruit more than one robot in order to complete a task. Any less than this amounts simply to an improvement in efficiency. Now that a good understanding of the minimal requirement for generating more efficient algorithms has been established, it is advantageous to undertake a study of the tradeoff between the efficiency of the collective, and the design complexity. This may yield optimally complex and efficient designs, using a minimum of collective processing, but yielding a maximally efficient collective.

Acknowledgements

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