Planning trajectories and address swaps to fabricate mobility traces

1 Motivation

This following attempts to provide definitions for a series of problems discussed by Amir Herzberg, Bing Wang, and Dylan Shell on Monday, 27 January 2025, in Amir's office at UConn. The basic idea is to have robots modify this network addresses so that an external observer, snooping on their communications, might be misled.

2 Basic formulation

Environment and robots A collection of n mobile robots populate in a planar environment containing a (potentially empty) finite set of disjoint obstacles $\mathcal{O} = O_1 \cup O_2 \cup \cdots \cup O_k$. Each robot's position is modeled as a point within $\mathscr{W} \coloneqq \mathrm{cl}(\mathbb{R}^2 \setminus \mathcal{O}) \subseteq \mathbb{R}^2$. We examine operation within a finite window of time, *viz.* $t \in \mathbb{T} \coloneqq [0, T]$ during which robots move in $\mathscr{W}$. Robots will execute continuous trajectories with speed bounded by $v_{\max} \in \mathbb{R}_{>0}$, so it is useful to define the following family of trajectories:

$$\mathcal{T} \coloneqq \left\{ \psi : \mathbb{T} \to \mathscr{W} \;\middle|\; \psi \text{ is of class } C^0 \text{ and } |\psi'(t)| \leq v_{\max}, \forall t \in \mathbb{T} \right\}. \tag{1}$$

Letting $[1..n]$ be shorthand for the set of indices $\{1, \ldots, n\}$, for each robot $i \in [1..n]$, denote its position by $\mathbf{x}_i \in \mathcal{T}$.

Identifiers and address assignments We use the subscript (as above, with $i \in [1..n]$) to distinguish, by an index, each physically distinct agent. In addition, we will assign an *address* from a space of identifiers $\mathcal{I} \coloneqq \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots\}$, where we require $|\mathcal{I}| \geq n$. A *static address assignment* is an injective map $m : [1..n] \to \mathcal{I}$. We will be more interested in instances where the assignment changes with time, hence, we consider a *dynamic address assignment* to be a map $z : \mathbb{T} \times [1..n] \to \mathcal{I}$, where, for each $t \in \mathbb{T}$, z_t gives a static address assignment. Let $\mathcal{A} \coloneqq \{z \mid z \text{ is a dynamic address assignment}\}$.

A key aspect of the model is that some measurement apparatus will employ a spatio-temporal triangulation process that, crucially, only has access to trajectories when associated with addresses; the following definitions help express this.

Definition 1 (Portrayal and fabrication)**.** A *portrayal* is a map from $\mathcal{I} \times \mathbb{T}$ to $\mathscr{W} \cup \{\perp\}$. Given trajectories $(\mathbf{x}_i)_{i \in [0..n]} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, for dynamic address assignment z, the *z-fabrication* is:

$$\Theta^{z}_{(\mathbf{x}_i)}(a, t) \coloneqq \begin{cases} \mathbf{x}_\alpha(t) & \text{if } z(t, \alpha) = a, \\ \perp & \text{otherwise.} \end{cases} \tag{2}$$

Because assignment $z(t, \cdot)$ is injective at fixed times, the resulting fabrication $\Theta^{z}_{(\mathbf{x}_i)}$ is a well-defined portrayal.

Here, the '$\perp$' symbol expresses absence of any associated trajectory: e.g., for portrayal χ, if $\chi(13, 7\frac{1}{4}) = \perp$ then this represents the fact that the portrayal includes no robot at time $t = 7.25$ with address 13.

Measurement (e.g., spatio-temporal triangulation) produce or capture data of the following form.

Definition 2 (Tracking trace)**.** An external entity collects *a tracking trace*, that is, a collection of data in the form of triples $\{\langle a_1, P_1, T_1\rangle, \langle a_2, P_2, T_2\rangle, \langle a_3, P_3, T_3\rangle \dots\}$ where each $a_i \in \mathcal{I}$, $P_i \subseteq \mathscr{W}$, and $T_i \subseteq \mathbb{T}$.

Simple example: If observations are made of an agent with address 42, precisely at times $t = 3$, $t = 8$, and $t = 13$, which started at the global origin $(0,0)$ and moved with constant speed in the horizontal direction, we could obtain trace

$$\mathscr{T}_0 = \left\{\langle 42, \left\{\left(6,0\right)^\intercal\right\}, \{3\}\rangle,\ \ \langle 42, \left\{\left(16,0\right)^\intercal\right\}, \{8\}\rangle,\ \ \langle 42, \left\{\left(26,0\right)^\intercal\right\}, \{13\}\rangle\right\}.$$

Example with uncertainty: Suppose an agent with address 51 is localized only imprecisely to within a unit-sized spatial disk. Then we might obtain the following trace, assuming detection was at times $t = 10 \pm \frac{1}{8}$ and $t = 21 \pm \frac{1}{2}$, owing to an inexact clock:

$$\mathscr{T}_1 = \left\{\langle 51, \left\{\left(x,y\right)^\intercal \middle| (x-2)^2 + y^2 \le 1\right\}, \left[9\tfrac{7}{8}, 10\tfrac{1}{8}\right]\rangle,\ \ \langle 51, \left\{\left(x,y\right)^\intercal \middle| (x-5)^2 + (y-3)^2 \le 1\right\}, \left[20\tfrac{1}{2}, 21\tfrac{1}{2}\right]\rangle\right\}.$$

These two example traces might correspond to observers with different capabilities of detection and localization. To be suitably general, we introduce the following definition.

Definition 3 (Observer)**.** An *observer* is an operator $\mathbf{M}$ that, given a portrayal, produces a tracking trace that is consistent with the portrayal, *viz.* if $\chi(\cdot,\cdot) \mapsto \mathbf{M}(\chi) = \{\langle a_1, P_1, T_1\rangle, \langle a_2, P_2, T_2\rangle, \langle a_3, P_3, T_3\rangle, \dots\}$ then $\langle a_k, P_k, T_k\rangle \in \mathbf{M}(\chi) \implies \exists t \in T_k : \chi(a_k, t) \in P_k$.

Observer strength: For any two observers, we will write $\mathbf{M}_1 \preccurlyeq \mathbf{M}_2$ if, for any portrayal $\chi : \mathcal{I} \times \mathbb{T} \to \mathscr{W} \cup \{\perp\}$, $\forall \langle a, P, T\rangle \in \mathbf{M}_1(\chi), \exists \langle a, P', T'\rangle \in \mathbf{M}_2(\chi)$ with $P' \subseteq P$ and $T' \subseteq T$. This yields a partial order on observers and, hence, a lattice with the observer which always produces the empty trace at the bottom. We will say $\mathbf{M}_1$ is weaker than $\mathbf{M}_2$ when $\mathbf{M}_1 \preccurlyeq \mathbf{M}_2$.

Easy examples: Imagine an observer that processes portrayals at integer times, sampling locations and reporting only locations within some rectangular region. An observer that samples at a subset of those times would be weaker. So too, an observer that considers a sub-region within the rectangle. If locations are imprecise (e.g., represented by circles), then weaker observers would be less precise (e.g., larger circles); similarly, weaker observers might include larger T_k sets.

Definition 4 (Implausiblity)**.** Given address $\bar{a} \in \mathcal{I}$, trajectory $\bar{\mathbf{x}} \in \mathcal{T}$, and trace $\mathscr{T}_0$, the triple $(\bar{a}, \bar{\mathbf{x}}, \mathscr{T}_0)$ is *implausible* if and only if there is some $\langle \bar{a}, P_k, T_k\rangle \in \mathscr{T}_0$ where $\bar{\mathbf{x}}(T_k) \cap P_k = \varnothing$. A triple is *plausible* if it is not implausible.

Intuitively, implausibility means trace $\mathscr{T}_0$ could not have been generated by an agent with static address $\bar{a}$ moving along $\mathbf{x}$.

3 Questions

These questions relate trajectories to traces generated by observers via construction of a fabrication — through dynamic address assignment — in order to plausibly exhibit some behavior.

Question 1 (Plausibility via address assignment)**.** Let some observer $\mathbf{M}$ be given. For a desired trajectory $\bar{\mathbf{x}} \in \mathcal{T}$ and an address $\bar{a} \in \mathcal{I}$, and some fixed trajectories for robots $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, find a dynamic address assignment z such that $\left(\bar{a}, \bar{\mathbf{x}}, \mathbf{M} \circ \Theta^z_{(\mathbf{x}_i)}\right)$ is plausible if and only if one exists.

A *trajectory modification cost* is a function $J : \mathcal{T}^n \times \mathcal{T}^n \to \mathbb{R} \cup \{\infty\}$ that gives a scalar cost to changing the series of trajectories. The value $J(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \mathbf{x}_1, \ldots, \mathbf{x}_n)$ denotes the work necessary to chage trajectories $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ into $\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n$. It will depend on the specific context. For instance, we might require the end-points remain fixed (giving a cost ∞ to any modification that violates this). Or we might require that the same set of end-points are reached, etc.

Question 2 (Plausible trajectory perturbation)**.** Let some observer $\mathbf{M}$ and a trajectory modification cost J be given. For any desired trajectory $\bar{\mathbf{x}} \in \mathcal{T}$ and address $\bar{a} \in \mathcal{I}$, and some initial trajectories for robots $\mathbf{x}_1, \ldots, \mathbf{x}_n$, find both address assignment $z^\star$ and a modified set of trajectories $\mathbf{x}^\star_1, \ldots, \mathbf{x}^\star_n$:

$$(\mathbf{x}^\star_1, \ldots, \mathbf{x}^\star_n; z^\star) = \underset{\substack{(\mathbf{x}'_1, \ldots, \mathbf{x}'_n) \in \mathcal{T}^n \\ z' \in \mathcal{A}}}{\arg\min} J(\mathbf{x}'_1, \ldots, \mathbf{x}'_n; \mathbf{x}_1, \ldots, \mathbf{x}_n) \text{ subject to } \left(\bar{a}, \bar{\mathbf{x}}, \mathbf{M} \circ \Theta^{z'}_{(\mathbf{x}'_i)}\right) \text{ being plausible.}$$

Question 3 (Strongest observer fooled)**.** Given a desired trajectory $\bar{\mathbf{x}} \in \mathcal{T}$, an address $\bar{a} \in \mathcal{I}$, fixed trajectories $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, and a dynamic address assignment z, find an observer $\mathbf{M}_\vee$, not necessarily unique, such that $\left(\bar{a}, \bar{\mathbf{x}}, \mathbf{M}_\vee \circ \Theta^z_{(\mathbf{x}_i)}\right)$ is plausible, and $\mathbf{M}_\vee \npreceq \mathbf{M}$ for every other $\mathbf{M}$ with plausible $\left(\bar{a}, \bar{\mathbf{x}}, \mathbf{M} \circ \Theta^z_{(\mathbf{x}_i)}\right)$.

All of the above questions might be generalized, obviously, to consider multiple desired trajectories. Also, many other variants are possible: for instance, Question 2 might impose costs for different address assignments (e.g., more frequent changes, or more drastic ones being more costly).